

## time series analysis

The analysis of economic time series is central to a wide range of applications, including business cycle measurement, financial risk management, policy analysis based on structural dynamic econometric models, and forecasting. This article provides an overview of the problems of specification, estimation and inference in linear stationary and ergodic time series models as well as non-stationary models, the prediction of future values of a time series and the extraction of its underlying components. Particular attention is devoted to recent advances in multiple time series modelling, the pitfalls and opportunities of working with highly persistent data, and models of non-linear dependence.

Any series of observations ordered along a single dimension, such as time, may be thought of as a time series. The emphasis in time series analysis is on studying the dependence among observations at different points in time. What distinguishes time series analysis from general multivariate analysis is precisely the temporal order imposed on the observations. Many economic variables, such as GNP and its components, price indices, sales, and stock returns are observed over time. In addition to being interested in the contemporaneous relationships among such variables, we are often concerned with relationships between their current and past values, that is, relationships over time.

The study of time series of, for example, astronomical observations predates recorded history. Early writers on economic subjects occasionally made explicit reference to astronomy as the source of their ideas. For example, Cournot (1838) stressed that, as in astronomy, it is necessary to recognize the *secular* variations which are independent of the periodic variations. Similarly, Jevons (1884) remarked that his study of short-term fluctuations used the methods of astronomy and meteorology. During the 19th century interest in, and analysis of, social and economic time series evolved into a new field of study independent of developments in astronomy and meteorology (see Nerlove, Grether and Carvalho, 1979, pp. 1–21, for a historical survey).

Harmonic analysis is one of the earliest methods of analysing time series thought to exhibit some form of periodicity. In this type of analysis, the time series, or some simple transformation of it, is assumed to be the result of the superposition of sine and cosine waves of different frequencies. However, since summing a finite number of such strictly periodic functions always results in a perfectly periodic series, which is seldom observed in practice, one usually allows for an additive stochastic component, sometimes called 'noise'. Thus, an observer must confront the problem of searching for 'hidden periodicities' in the data, that is, the unknown frequencies and amplitudes of sinusoidal fluctuations hidden amidst noise. An early method for this purpose is *periodogram analysis*, suggested by Stokes (1879) and used by Schuster (1898) to analyse sunspot data and later by others, principally William Beveridge (1921; 1922), to analyse economic time series.

Spectral analysis is a modernized version of periodogram analysis modified to take account of the stochastic nature of the entire time series, not just the noise component. If it is assumed that economic time series are fully stochastic, it follows that the older periodogram technique is inappropriate and that considerable difficulties in the interpretation of the periodograms of economic series may be encountered.

At the time when harmonic analysis proved to be inadequate for the analysis of economic and social time series, another way of characterizing

such series was suggested by the Russian statistician and economist, Eugen Slutsky (1927), and by the British statistician, G.U. Yule (1921; 1926; 1927). Slutsky and Yule showed that, if we begin with a series of purely random numbers and then take sums or differences, weighted or unweighted, of such numbers, the new series so produced has many of the apparent cyclic properties that were thought at the time to characterize economic and other time series. Such sums or differences of purely random numbers and sums or differences of the resulting series form the basis for the class of autoregressive moving-average (ARMA) processes which are used for modelling many kinds of time series. ARMA models are examples of time domain representations of time series. Although the latter may look very different from spectral representations of time series, there is a one-to-one mapping between time domain analysis and spectral analysis. Which approach is preferred in practice is a matter only of convenience. The choice is often determined by the transparency with which a given question can be answered. The remainder of this article explores these two complementary approaches to the analysis of economic time series.

## 1. Basic theory

### 1.1. Stationarity and ergodicity of time series processes

Consider a random variable  $x_t$  where  $t \in N$ , the set of integers; the infinite vector  $\{x_t, t \in N\}$  is called a discrete time series. Let  $M$  denote a subset of  $T$  consecutive elements of  $N$ . The distribution of the finite dimensional vector  $\{x_t, t \in M\}$  is a well-defined multivariate distribution function,  $F_M(\cdot)$ . The time series  $\{x_t, t \in N\}$  is said to be *strictly stationary* if, for any finite subset  $M$  of  $N$  and any integer  $\tau$ , the distribution function of  $\{x_t, t \in M + \tau\}$  is the same as the distribution function of  $\{x_t, t \in M\}$ . In other words, the joint distribution function of the finite vector of observations on  $x_t$  is invariant with respect to the origin from which time is measured. All the unconditional moments of the distribution function, if they exist, are independent of the index  $t$ ; in particular,

$$\begin{aligned} E(x_t) &= \mu \\ \gamma(\tau) &= E[x_t - \mu][x_{t+\tau} - \mu], \end{aligned} \tag{1}$$

where  $\gamma(\tau)$  is the autocovariance function and depends only on the difference in indices,  $\tau$ . Time-series processes for which (1) holds, but which are not necessarily strictly stationary according to the definition above, are said to be weakly *stationary*, *covariance stationary*, or *stationary to the second order*. Time-series processes for which  $F_M(\cdot)$  is multivariate normal for any subset  $M$  of  $N$  are called *Gaussian processes*. For Gaussian processes covariance stationarity implies strict stationarity.

In practice, we usually observe only one realization of a finite subset of the time series of interest, corresponding to one of the many possible draws of length  $T$  from  $F_M(\cdot)$ . The question is whether the moments of  $x_t$  may be inferred from one such realization; for example, from the time averages of sums (or sums of products) of the observed values of a time series. If the process is what is known as *ergodic*, time averages of functions of the observations on the time series at  $T$  time points converge in mean square to the corresponding population expectations of  $x_t$  across alternative draws, as  $T \rightarrow \infty$  (Priestley, 1981, pp. 340–3; Doob, 1953, p. 465). It is possible for a process to be stationary, yet not ergodic. Consider, for example, the process

$x_t^{(i)} = \eta^{(i)} + \varepsilon_t$ , where  $x_t^{(i)}$  denotes the  $i$ th draw for observation  $x_t$  from the universe of all possible draws for  $x_t$ . Suppose that  $\eta^{(i)} \sim N(0, \lambda^2)$  is the mean of the  $i$ th draw and that  $\varepsilon_t \sim N(0, \sigma^2)$  is independent of  $\eta^{(i)}$ . This process is clearly stationary in that the probability limit of the ensemble average is zero, yet the time average  $\sum_{t=1}^T x_t^{(i)}/T = \eta^{(i)} + \sum_{t=1}^T \varepsilon_t/T$  converges to  $\eta^{(i)}$  rather than zero, thus violating ergodicity.

### 1.2. The Wold decomposition and general linear processes

Let  $\{\varepsilon_t\}$  be one element of a time series of serially uncorrelated, identically distributed random variables with zero mean and variance  $\sigma^2$ . Then the infinite, one-sided moving average (MA) process

$$x_t = \sum_{j=0}^{\infty} b_j \varepsilon_{t-j}, \quad (2)$$

where  $b_0 = 1$  and  $\sum_{j=0}^{\infty} b_j^2 < \infty$ , is also a well-defined stationary process with mean 0 and variance  $\sigma^2 \sum_{j=0}^{\infty} b_j^2$ . Processes of this form and, more generally, processes based on an infinite two-sided MA of the same form are called *linear processes*, are always ergodic, and play a key role in time series analysis (Hannan, 1970).

The importance of the process (2) is underscored by the Wold decomposition theorem (Wold, 1938), which states that any weakly stationary process may be decomposed into two mutually uncorrelated component processes, one an infinite one-sided MA of the form (2) and the other a so-called linearly deterministic process, future values of which can be predicted exactly by some linear function of past observations. The linearly deterministic component is non-ergodic.

## 2. Linear processes in time and frequency domains

### 2.1. Autocovariance and autocovariance generating functions

The autocovariance function of a stationary process, defined in (1) above, or its matrix generalization for vector processes, provides the basic representation of time dependence for weakly stationary processes. For the stationary process defined in (2), it is

$$\gamma(\tau) = \sigma^2 \sum_{j=0}^{\infty} b_j b_{j+\tau}. \quad (3)$$

Let  $z$  denote a complex scalar. Then the autocovariance generating transform is defined as

$$g(z) = \sum_{-\infty}^{\infty} \gamma(\tau) z^\tau \quad (4)$$

in whatever region of the complex plane the series on the right-hand side converges. If the series  $\{x_t\}$  is covariance stationary, convergence will occur in an annulus about the unit circle. The autocovariance generating transform for the one-sided MA process defined in (2) is

$$g(z) = \sigma^2 B(z) B(z^{-1}) \quad (5)$$

where

$$B(z) = \sum_{k=0}^{\infty} b_k z^k.$$

If  $B(z)$  has no zeros on the unit circle, the process defined in (2) is invertible and also has an infinite-order autoregressive (AR) representation as

$$A(L)x_t = \varepsilon_t, \quad (6)$$

where  $L$  is the lag operator such that  $L^j x_t = x_{t-j}$  and  $A(L) = a_0 + a_1 L + a_2 L^2 + \dots$

So-called ARMA processes have an autocovariance generating transform which is a rational function of  $z$ . If the ARMA process is both stationary and invertible,  $g(z)$  may be written as

$$G(z) = \frac{P(z)P(z^{-1})}{Q(z)Q(z^{-1})} = \sigma^2 \frac{\prod_{k=1}^m (1 - \beta_k z)(1 - \beta_k z^{-1})}{\prod_{j=1}^n (1 - \alpha_j z)(1 - \alpha_j z^{-1})} \quad (7)$$

where  $|\beta_k|, |\alpha_j| < 1 \forall j, k$ . Then the corresponding ARMA model is

$$Q(L)x_t = P(L)\varepsilon_t, \quad (8)$$

where

$$Q(L) = \prod_{j=1}^n (1 - \alpha_j L) \text{ and } P(L) = \prod_{k=1}^m (1 - \beta_k L).$$

## 2.2. Spectral density functions

If the value of  $z$  lies on the complex unit circle, it follows that  $z = e^{-i\lambda}$ , where  $i = \sqrt{-1}$  and  $-\pi \leq \lambda \leq \pi$ . Substituting for  $z$  in the autocovariance generating transform (5) and dividing by  $2\pi$ , we obtain the *spectral density function* of a linearly non-deterministic stationary process  $\{x_t\}$  in terms of the frequency  $\lambda$ :

$$\begin{aligned} f(\lambda) &= (1/2\pi)g(e^{i\lambda}) = (\sigma^2/2\pi)B(e^{i\lambda})B(e^{-i\lambda}) \\ &= (1/2\pi) \sum_{-\infty}^{\infty} \gamma(\tau) e^{-i\lambda\tau}, \quad -\pi \leq \lambda < \pi. \end{aligned} \quad (9)$$

Thus, the spectral density function is the Fourier transform of the autocovariance function. It can be shown that for a process with absolutely summable autocovariances the spectral density function exists and can be used to compute all of the autocovariances, so the same time series can be characterized equivalently in terms of the autocovariance function in the time-domain or in terms of the spectral density function in the frequency domain.

The spectral density function for a linearly non-deterministic, stationary, real-valued time series is a real-valued, non-negative function, symmetric about the origin, defined in the interval  $[-\pi, \pi]$ :

$$f(\lambda) = (1/2\pi) \left[ \gamma(0) + 2 \sum_{\tau=1}^{\infty} \gamma(\tau) \cos \lambda\tau \right]. \quad (10)$$

Moreover,

$$E(x_t - \mu)^2 = \int_{-\pi}^{\pi} f(\lambda) d\lambda, \quad (11)$$

so that the spectral density function is a frequency-band decomposition of the variance of  $\{x_t\}$ .

When the process generating  $\{x_t\}$  is merely stationary, that is, when  $\{x_t\}$  may have a linearly deterministic component, the spectral density function is

$$f(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda\tau} dF(\lambda), \quad (12)$$

where  $F(\lambda)$  is a distribution function (Doob, 1953, p. 488). Note that deterministic seasonal effects, for example, may cause a jump in the spectral distribution function.

The autocovariance function, its generating transform and the spectral distribution function all have natural generalizations to the multivariate case, in which  $\{x_t\}$  can be thought of as a vector of time-series processes.

The estimation and analysis of spectral density and distribution functions play an important role in all forms of time-series analysis. More detailed treatments are Doob (1953), Fishman (1969), Koopmans (1974), Fuller (1976), Nerlove, Grether and Carvalho (1979, ch. 3), and Priestley (1981).

### 2.3. Unobserved components (UC) models

In the statistical literature dealing with the analysis of economic time series it is common practice to classify the types of movements that characterize a time series as trend, cyclical, seasonal, and irregular components. The idea that a time series may best be viewed as being composed of several unobserved components is by no means universal, but it plays a fundamental role in many applications, for example, the choice of methods for seasonal adjustment. Nerlove, Grether and Carvalho (1979, ch. 1) review the history of the idea of unobserved components in economics from its origin early in the 19th century.

In the 1960s, Nerlove (1964; 1965; 1967) and Granger (1966) suggested that the typical spectral shape of many economic time series could be accounted for by the superposition of two or more independent components with specified properties. There are basically two approaches to the formulation of UC models: Theil and Wage (1964) and Nerlove and Wage (1964), Nerlove (1967) and Grether and Nerlove (1970) choose the form of components in such a way as to replicate the typical spectral shape of the series which represents their superposition. For example, let  $T_t$  represent the trend component,  $C_t$  the cyclical,  $S_t$  the seasonal, and  $I_t$  the irregular of a monthly time series; then the observed series can be represented as

$$y_t = T_t + C_t + S_t + I_t, \quad (13)$$

where

$$\begin{aligned} T_t &= a_0 + a_1 t + a_2 t^2 + \dots + a_p t^p, \\ C_t &= \frac{1 + \beta_1 L + \beta_2 L^2}{(1 - \alpha_1 L)(1 - \alpha_2 L)} \varepsilon_{1t}, \\ S_t &= \frac{1 + \beta_3 L + \beta_4 L^2}{1 - \gamma L^{12}} \varepsilon_{2t}, \\ I_t &= \varepsilon_{3t}, \end{aligned}$$

and  $\varepsilon_{1t}$ ,  $\varepsilon_{2t}$ , and  $\varepsilon_{3t}$  are i.i.d. normal variables with variances  $\sigma_{11}$ ,  $\sigma_{22}$ , and  $\sigma_{33}$ ,

respectively. This approach has been carried forward by Harvey (1984), Harvey and Peters (1990) and Harvey and Todd (1984).

An alternative approach is to derive the components of the UC model from a well-fitting ARMA model (obtained after suitably transforming the data), given sufficient a priori identifying restrictions on the spectral properties of the components. See Box, Hillmer and Tiao (1978), Pierce (1978; 1979), Burman (1980), Hillmer and Tiao (1982), Hillmer, Bell and Tiao (1983), Bell and Hillmer (1984), Burrige and Wallis (1985), and Maravall (1981; 1984). The basis of this procedure is the fact that every stationary UC model, or the stationary part of every UC model, has an equivalent ARMA form, the so-called canonical form of the UC model (Nerlove and Wage, 1964; Nerlove, Grether and Carvalho, 1979, ch. 4).

### 3. Specification, estimation, inference and prediction

#### 3.1. Autocovariance and spectral density functions

Suppose we have a finite number of observations of a realization of the process generating the time series, say  $x_1, \dots, x_T$ . For expository purposes it is assumed that all deterministic components of  $x_t$  have been removed. If  $\mu$  is unknown, this may be accomplished by subtracting the sample mean of the time series observations from the data prior to the analysis. For a zero mean series  $x_t$  there are basically two ways of estimating  $\gamma(\tau)$  defined in (1): the first is the biased estimator

$$c(\tau) = (1/T) \sum_{t=1}^{T-|\tau|} x_t x_{t+|\tau|}, \quad \tau = 0, \pm 1, \dots, \pm M, M \leq (T-1). \quad (14)$$

The second is the unbiased estimator

$$\tilde{c}(\tau) = [1/(T-|\tau|)] \sum_{t=1}^{T-|\tau|} x_t x_{t+|\tau|}, \quad \tau = 0, \pm 1, \dots, \pm M, \quad M \leq T-1. \quad (15)$$

Although  $c(\tau)$  is biased in finite samples, it is asymptotically unbiased. The key difference between  $c(\tau)$  and  $\tilde{c}(\tau)$  is that  $c(\tau)$  is a positive definite function of  $\tau$  whereas  $\tilde{c}(\tau)$  is not (Parzen, 1961, p. 981). The variance and covariances of the estimated autocovariances are derived, *inter alia*, by Hannan (1960), and Anderson (1971). As  $T \rightarrow \infty$ , both tend to zero, as the estimates are asymptotically uncorrelated and consistent. However,

$$E[c(\tau) - Ec(\tau)]^2 / E[c(\tau)] \rightarrow \infty \quad \text{as } \tau/T \rightarrow 1. \quad (16)$$

This property accounts for the failure of the estimated autocorrelation function

$$r(\tau) = c(\tau)/c(0) \quad (17)$$

to damp down as  $\tau \rightarrow \infty$ , as it should for a stationary, linearly non-deterministic process (Hannan, 1960, p. 43).

A 'natural' estimator of the spectral density function is obtained by replacing  $\gamma(\tau)$  in (10) by  $c(\tau)$  or  $\tilde{c}(\tau)$ . The resulting estimator is proportional, at each frequency, to a sample quantity called the periodogram:

$$I_T(\lambda) = (2/T) \left| \sum_{t=1}^T e^{i\lambda t} x_t \right|^2 \quad (18)$$

usually evaluated at the equi-spaced frequencies

$$\lambda = 2k\pi/T, \quad k = 1, 2, \dots, [T/2] \quad (19)$$

in the interval  $[0, \pi]$ . Although, for a stationary, nonlinearly deterministic process, the periodogram ordinates are asymptotically unbiased estimates of the spectral densities at the corresponding frequencies, they are not consistent estimates; moreover, the correlation between adjacent periodogram ordinates tends to zero with increasing sample size. The result is that the periodogram presents a jagged appearance which is increasingly difficult to interpret as more data become available.

In order to obtain consistent estimates of the spectral density function at specific frequencies, it is common practice to weight the periodogram ordinates over the frequency range or to form weighted averages of the autocovariances at different lags. There is a substantial literature on the subject. The weights are called a ‘spectral window’. Essentially the idea is to reduce the variance of the estimate of an average spectral density around a particular frequency by averaging periodogram ordinates which are asymptotically unbiased and independently distributed estimates of the corresponding ordinates of the spectral density function. Related weights can also be applied to the estimated autocovariances which are substituted in (10); this weighting system is called a ‘lag window’.

Naturally the sampling properties of the spectral estimates depend on the nature of the ‘window’ used to obtain consistency (see Priestley, 1981, pp. 432–94 for further discussion). Regardless of the choice of window, the ‘bandwidth’ used in constructing the window must decrease at a suitable rate as the sample size grows. In the spectral window approach, this means that the window width must decrease at a slower rate than the sample size. In the lag window approach, this means that the number of included autocovariances must increase at a slower rate than the sample size.

### 3.2. ARMA models

The autocovariance function and the spectral density function for a time series represent nonparametric approaches to describing the data. An alternative approach is to specify and estimate a parametric ARMA model for  $x_t$ . This approach involves choosing the orders of the polynomials  $P$  and  $Q$  in (7) and (8) and perhaps also specifying that one or more coefficients are zero or placing other restrictions on  $P$  and  $Q$ . The problem then becomes one of estimating the parameters of the model.

Despite the poor statistical properties of the estimated autocovariance function and a related function called the partial autocorrelation function, these are sometimes used to specify the orders of the polynomials  $P$  and  $Q$ . An alternative approach is to select the model that minimizes the value of information-theoretic criteria of the form

$$\text{IC}(i) = \log(\hat{\sigma}_i^2) + k_i c_T, \quad (20)$$

where  $k_i$  refers to the number of estimated parameters in the candidate models  $i = 1, \dots, M$ , and  $\hat{\sigma}_i^2$  to the corresponding maximum likelihood estimate of the residual variance. Such criteria incorporate a trade-off between the fit of a model and its degree of parsimony. That trade-off depends on the penalty term  $c_i$  (Akaike, 1970; 1974; Schwarz, 1978). There is no universally accepted choice for  $c_T$ . For  $C_T = 2/T$  expression (20) reduces to the Akaike information criterion (AIC), for example, and for  $C_T = \ln(T)/T$  to the Schwarz information criterion (SIC). The asymptotic properties of alterna-

tive criteria will depend on the objective of the user and the class of models considered.

Given the orders of the AR and MA components, a variety of maximum likelihood or approximate maximum likelihood methods are available to estimate the model parameters. Newbold (1974) shows that, if  $x_t$  is characterized by (8) with  $\varepsilon_t \sim NID(0, \sigma^2)$ , then the exact likelihood function for the parameters of  $P(\cdot)$  and  $Q(\cdot)$  is such that the maximum likelihood estimates of the parameters and the least-squares (LS) estimates (in general highly nonlinear) are asymptotically identical. Only in the case of a pure AR model are the estimates linear conditional on the initial observations. Several approximations have been discussed (Box and Jenkins, 1970; Granger and Newbold, 1977; Nerlove, Grether and Carvalho, 1979, pp. 121–5).

Exact maximum likelihood estimation of ARMA models has been discussed by, inter alia, Newbold (1974), Anderson (1977), Ansley (1979), and Harvey (1981). Following Schweppe (1965), Harvey suggests the use of the Kalman filter (1960) to obtain the value of the exact-likelihood function, which may be maximized by numerical methods. The Kalman filter approach is easily adapted to the estimation of UC models in the time domain.

An alternative to exact or approximate maximum-likelihood estimation in the time domain was suggested by Hannan (1969). Estimates may be obtained by maximizing an approximate likelihood function based on the asymptotic distribution of the periodogram ordinates defined in (18). These are asymptotically independently distributed (Brillinger, 1975, p. 95), and the random variables  $2I_t(\lambda)/f(\lambda)$  have an asymptotic  $\chi^2$  distribution with two degrees of freedom (Koopmans, 1974, pp. 260–5). This means that the asymptotic distribution of the observations,  $\{x_1, \dots, x_T\}$  is proportional to

$$\prod_{j=0}^{[T/2]} [1/f(\lambda_j)] \exp[-I(\lambda_j)/f(\lambda_j)] \quad (21)$$

where  $\lambda_j = 2j\pi/T, j = 0, \dots, [T/2]$ , are the equi-spaced frequencies in the interval  $[0, \pi]$  at which the periodogram is evaluated (Nerlove, Grether and Carvalho, 1979, pp. 132–6). Since the true spectral density  $f(\lambda)$  depends on the parameters characterizing the process, this asymptotic distribution may be interpreted as a likelihood function. Frequency domain methods, as these are called, may easily be applied in the case of UC models.

Whether approximate or exact maximum-likelihood estimation methods are employed, inference may be based on the usual criteria related to the likelihood function. Unfortunately, serious difficulties may be encountered in applying the asymptotic theory, since the small sample distribution of the maximum likelihood estimator may differ greatly from the limiting distribution in important cases (Sargan and Bhargava, 1983; Anderson and Takemura, 1986).

### 3.3. Prediction and extraction

The problem of prediction is essentially the estimation of an unknown future value of the time series itself; the problem of extraction, best viewed in the context of UC models described in section 2.3, is to estimate the value of one of the unobserved components at a particular point in time, not necessarily in the future. Problems of trend extraction and seasonal adjustment may be viewed in this way (Grether and Nerlove, 1970). How the prediction (or extraction) problem is approached depends on whether we are assumed to have an infinite past history and, if not, whether the parameters of the

process generating the time series are assumed to be known. In practice, of course, an infinite past history is never available, but a very long history is nearly equivalent if the process is stationary or can be transformed to stationarity. It is common, as well, to restrict attention to linear predictors, which involves no loss of generality if the processes considered are Gaussian and little loss if merely linear. To devise a theory of optimal prediction or extraction requires some criterion by which to measure the accuracy of a particular candidate. The most common choice is the minimum mean-square error (MMSE) criterion, which is also the conditional expectation of the unknown quantity. For a discussion of alternative loss functions see Granger (1969) and Christoffersen and Diebold (1996; 1997).

The theory of optimal prediction and extraction due to Kolmogorov (1941) and Wiener (1949) and elaborated by Whittle (1963) for discrete processes assumes a possibly infinite past history and known parameters. As a special case of the Wiener–Kolmogorov theory for non-deterministic, stationary processes, consider the linear process defined by (2). Since the  $\varepsilon_t$  are i.i.d. with zero mean and variance  $\sigma^2$ , it is apparent that the conditional expectation of  $x_{t+v}$ , given all innovations from the infinite past to  $t$ , is

$$\hat{x}_{t+v} = b_v \varepsilon_t + b_{v+1} \varepsilon_{t+1} + \dots \quad (22)$$

Of course, even if the parameters  $b_j, j = 0, 1, \dots$ , are assumed to be known, the series  $\{\varepsilon_t\}$  is not directly observable. The  $\varepsilon_t$ 's are sometimes called the *innovations* of the process, since it is easy to show that  $\varepsilon_{t+1} = x_{t+1} - \hat{x}_{t+1}$  is the one-step ahead prediction error. If the process is invertible, it has the autoregressive representation (6) and so can be expressed solely in terms of the, generally infinite-order, autoregression

$$\hat{x}_{t+v} = D(L)x_t, \quad (23)$$

where the generating transform of the coefficients of  $D$  is

$$D(z) = \frac{1}{B(z)} \left[ \frac{B(z)}{z^v} \right]_+.$$

The operator  $[\cdot]_+$  eliminates terms involving negative powers of  $z$ .

The problem of extraction is best viewed in the context of multiple time series; in general we wish to ‘predict’ one time series  $\{y_t\}$  from another related series  $\{x_t\}$ . It is not necessary that the series  $\{y_t\}$  actually be observed as long as its relationship to an observed series  $\{x_t\}$  can be described (Nerlove, Grether and Carvalho, 1979, ch. 5).

The Kalman filter approach to prediction and extraction (Kalman, 1960) is both more special and more general than the Wiener–Kolmogorov theory: attention is restricted to finite-dimensional parameter spaces and linear processes, but these processes need not be stationary. The parameters may vary with time, and we do not require an infinite past. This approach represents a powerful tool of practical time-series analysis and may be easily extended to multiple time series. A full discussion, however, requires a discussion of ‘state-space representation’ of time series processes and is beyond the scope of this entry (Harvey, 1989)

#### 4. Multiple time series analysis

A general treatment of multiple time series analysis is contained in Hannan (1970). The two-variable case will serve to illustrate the matter. Two stationary time series  $\{x_t\}$  and  $\{y_t\}$  are said to be jointly stationary if their joint distribution function does not depend on the origin from which time is

measured. Joint stationarity implies, but is not in general implied by, weak or covariance joint stationarity; that is,  $cov(x_t, y_s)$  is a function of  $s - t$  only. In this case the cross-covariance function is

$$\gamma_{yx}(\tau) = E[y_t - \mu_y][x_{t-\tau} - \mu_x], \quad (24)$$

where  $\mu_x = Ex_t$  and  $\mu_y = Ey_t$ . Note that  $\gamma_{yx}(\tau)$  and  $\gamma_{xy}(\tau)$  are, in general, different. The cross-covariance generating function is defined as

$$g_{yx}(z) = \sum_{-\infty}^{\infty} \gamma_{yx}(\tau) z^\tau \quad (25)$$

in that region of the complex plane in which the right-hand side of (25) converges. For two jointly stationary series this occurs in an annulus containing the unit circle. In this case, the cross-spectral density function is defined as

$$f_{yx}(\lambda) = (1/2\pi) g_{yx}(e^{i\lambda}). \quad (26)$$

Since  $\gamma_{yx}(\tau)$  and  $\gamma_{xy}(\tau)$  are not equal, the cross-spectral density function is complex valued and can be decomposed into a real part (the co-spectral density) and a complex part (the quadrature spectral density):

$$f_{yx}(\lambda) = c_{yx}(\lambda) + iq_{yx}(\lambda). \quad (27)$$

In polar form, the cross-spectral density may be written as

$$f_{yx}(\lambda) = \alpha_{yx}(\lambda) \exp[i\phi_{yx}(\lambda)], \quad (28)$$

where  $\alpha_{yx}(\lambda) = [c_{yx}^2(\lambda) + q_{yx}^2(\lambda)]^{1/2}$  is called the amplitude or gain, and where  $\phi_{yx}(\lambda) = \arctan\{-q_{yx}(\lambda)/c_{yx}(\lambda)\}$  is called the phase. Another useful magnitude is the coherence between the two series, defined as

$$\rho_{yx}(\lambda) = \frac{|f_{yx}(\lambda)|^2}{f_{xx}(\lambda)f_{yy}(\lambda)}, \quad (29)$$

which measures the squared correlation between  $y$  and  $x$  at a frequency  $\lambda$ . Clearly,  $\rho_{yx}(\lambda) = \rho_{xy}(\lambda)$ . Estimation of cross-spectral density functions and related quantities is discussed in Priestley (1981, pp. 692–712).

Often it is convenient to impose additional parametric structure in modelling multiple time series. The workhorse multiple time series model in econometrics has been the covariance-stationary  $K$ -dimensional vector autoregressive model, which may be viewed as a natural generalization of the univariate AR model discussed earlier:

$$A(L)x_t = \varepsilon_t \quad (30)$$

where  $A(L) = I_K - A_1L - \dots - A_pL^p$ . Here each variable in  $x_t$  is regressed on its own lags as well as lags of all other variables in  $x_t$  up to some pre-specified lag order  $p$ . This *vector autoregression* (VAR) can also be viewed as an approximation to a general linear process  $x_t$ , and may be estimated by LS.

Similarly, the formulation of ARMA and UC models discussed earlier may be extended to the multivariate case by interpreting the polynomials in the lag operator as matrix polynomials and by replacing the scalar random variables by vectors. Although these vector ARMA and UC models bear a superficial resemblance to the corresponding univariate ones, their structure is, in fact, much more complicated and gives rise to difficult identification problems. In the univariate case, we can formulate simple conditions under which a given covariance function identifies a unique ARMA or UC model, but in the multivariate case these conditions are no longer sufficient. Hannan

(1970; 1971) gives a complete treatment. State-space methods have also been employed to study the structure of multivariate ARMA models (Hannan, 1976; and, especially, 1979).

## 5. Unit roots, co-integration and long memory

Standard tools for time series analysis have been developed for processes that are covariance stationary or have been suitably transformed to achieve covariance stationarity by removing (or explicitly modelling) deterministic trends, structural breaks, and seasonal effects. The presence of a *unit root* in the autoregressive lag order polynomial of an ARMA process also violates the assumption of stationarity. Processes with a unit root are also called integrated of order one (or  $I(1)$  for short) because they become covariance-stationary only upon being differenced once. In general,  $I(d)$  processes must be differenced  $d$  times to render the process covariance-stationary.

The presence of unit roots has important implications for estimation and inference. When the scalar process  $x_t$  is  $I(1)$  the variance of  $x_t$  will be unbounded, model innovations will have permanent effects on the level of  $x_t$ , the autocorrelation function does not die out, and  $x_t$  will not revert to a long-run mean. Moreover, coefficients of  $I(1)$  regressors will have nonstandard asymptotic distributions, invalidating standard tools of inference.

The simplest example of an autoregressive integrated moving-average (ARIMA) process is the random walk process:  $x_t = x_{t-1} + \varepsilon_t$ . The potential pitfalls of regression analysis with  $I(1)$  data are best illustrated by the problem of regressing one independent random walk on another. In that case, it can be shown that  $R^2$  and  $\hat{\beta}$  will be random and that the usual  $t$ -statistic will diverge, giving rise to seemingly significant correlations between variables that are unrelated by construction. This *spurious regression* problem was first discussed by Yule (1926), further illustrated by Granger and Newbold (1974), and formally analyzed by Phillips (1986) and Phillips and Durlauf (1986). Similar problems arise in deterministically detrending  $I(1)$  series (Nelson and Kang, 1981; Durlauf and Phillips, 1988). Unbalanced regressions, that is, regressions in which the regressand is not of the same order of integration as the regressor, may also result in spurious inference. An exception to this rule is inference on coefficients of mean zero  $I(0)$  variables in regressions that include a constant term (Sims, Stock and Watson, 1991).

The standard response to dealing with  $I(1)$  data is to difference the data prior to the analysis. There is one important exception to this rule. There are situations in which several variables are individually  $I(1)$ , but share a common unit root component. In that case, a linear combination of these variables will be  $I(0)$ :

$$c'x_t = u_t \sim I(0), \quad c \neq 0 \quad (31)$$

where  $x_t$  denotes a  $K$ -dimensional vector of  $I(1)$  variables and  $c$  is a  $(K \times 1)$  parameter vector. In other words, these variables share a common stochastic trend. This phenomenon is known as *co-integration* (Granger, 1981; Engle and Granger, 1987) and  $c$  is known as the co-integrating vector. Clearly,  $c$  is not unique. It is common to normalize one element of  $c$  to unity. The LS estimator of  $c$  in (31) is consistent, but corrections for omitted dynamics are recommended (Stock and Watson, 1993; Phillips and Hansen, 1990). Co-integrating relationships have been used extensively in modelling long-run equilibrium relationships in economic data (Engle and Granger, 1991).

Variables that are co-integrated are linked by an *error correction* mechanism that prevents the integrated variables from drifting apart without

bound. Specifically, by the Granger representation theorem of Engle and Granger (1987), under some regularity conditions, any  $K$ -dimensional vector of co-integrated variables  $x_t$  can be represented as a vector error correction (VEC) model of the form:

$$\Delta x_t = \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} - \Pi x_{t-p} \quad (32)$$

where  $\Gamma_i$ ,  $i = 1, \dots, p-1$ , and  $\Pi \equiv BC$  are conformable coefficient matrices and  $\Delta$  denotes the first-difference operator. Model (32) allows for up to  $r$  co-integrating relationships where  $r$  is the rank of  $\Pi$ . For  $r = 0$ , the error correction term in model (32) drops out and the model reduces to a difference-stationary VAR. For  $r = K$ , all variables are  $I(0)$  and model (32) is equivalent to a stationary VAR in levels. Otherwise, there are  $0 < r < K$  common trends. If the  $(r \times K)$  matrix of co-integrating vectors,  $C$ , is known, the model in (32) reduces to

$$\Delta x_t = \sum_{i=1}^{p-1} \Gamma_i \Delta x_{t-i} - B z_{t-p} \quad (32')$$

where  $z_{t-p} \equiv C x_{t-p}$ , and the model may be estimated by LS; if only the rank  $r$  is known, the VEC model in (32) is commonly estimated by full information maximum likelihood methods (Johansen, 1995).

Starting with Nelson and Plosser (1982), a large literature has dealt with the problem of statistically discriminating between  $I(1)$  and  $I(0)$  models for economic data. Notwithstanding these efforts, it has remained difficult to detect reliably the existence of a unit root (or of co-integration). The problem is that in small samples highly persistent, yet stationary processes are observationally equivalent to exact unit root processes. It may seem that not much could hinge on this distinction then, but it can be shown that  $I(1)$  and  $I(0)$  specifications that fit the data about equally well may have very different statistical properties and economic implications (Rudebusch, 1993).

For processes with roots near-unity in many cases neither the traditional asymptotic theory for  $I(0)$  processes nor the alternative asymptotic theory for exact  $I(1)$  processes will provide a good small-sample approximation to the distribution of estimators and test statistics. An alternative approach is to model the dominant root,  $\rho$ , of the autoregressive lag order polynomial as *local-to-unity* in the sense that  $\rho = 1 - c/T$ ,  $c > 0$ . This asymptotic thought experiment gives rise to an alternative asymptotic approximation that in many cases provides a better small-sample approximation than imposing the order of integration or relying on unit root pretests (Stock, 1991; Elliott, 1998).

Stationary ARMA processes are ‘short memory’ processes in that their autocorrelation function dies out quickly. For large  $\tau$ , ARMA autocorrelations decay approximately geometrically, that is,  $\rho(\tau) \approx r^\tau$ , where  $r$  is a constant such that  $|r| < 1$ . In many applied contexts including volatility dynamics in asset returns, there is evidence that the autocorrelation function dies out much more slowly. This observation has motivated the development of the class of *fractionally integrated ARMA* (ARFIMA) models:

$$Q(L)(1-L)^d x_t = P(L)\varepsilon_t \quad (33)$$

where  $d$  is a real number, as opposed to an integer (Baillie, 1996). Stationarity and invertibility require  $|d| < 0.5$ , which can always be achieved by taking a suitable number of differences. The autocorrelation function of an ARFIMA process decays at a hyperbolic rate. For large  $\tau$ , we have  $\rho(\tau) \approx \tau^{2d-1}$ , where  $d < 1/2$  and  $d \neq 0$ . Such ‘long memory’ models may be estimated

by the two-step procedure of Geweke and Porter-Hudak (1983) or by maximum likelihood (Sowell, 1992; Baillie, Bollerslev and Mikkelsen, 1996). A detailed discussion including extensions to the notion of fractional co-integration is provided by Baillie (1996). Long memory may arise, for example, from infrequent stochastic regime changes (Diebold and Inoue, 2001) or from the aggregation of economic data (Granger, 1980; Chambers, 1998). Perhaps the most successful application of long-memory processes in economics has been work on modelling the volatility of asset prices and powers of asset returns, yielding new insights into the behaviour of markets and the pricing of financial risk.

## 6. Nonlinear time series models

The behaviour of many economic time series appears to change distinctly at irregular intervals, consistent with economic models that suggest the existence of floors and ceilings, buffer stocks, and regime switches in the data. This observation has given rise to a large literature dealing with nonlinear time series models. Nonlinear time series models still have a Wold representation with linearly unpredictable innovations, but these innovations are nevertheless dependent over time. This has important implications for forecasting and for the dynamic properties of the model. For example, the effects of innovations in nonlinear models will depend on the path of the time series and the size of the innovation, and may be asymmetric.

### 6.1. Nonlinear dynamics in the conditional mean

The increasing importance of nonlinear time series models in econometrics is best illustrated by two examples: hidden Markov chain models and smooth transition regression models of the conditional mean.

The idea of hidden Markov chains first attracted attention in econometrics in the context of *regime switching models* (Hamilton, 1989). The original motivation was that many economic time series appear to follow a different process during recession phases of the business cycle than during economic expansions. This type of regime-switching behaviour may be modelled in terms of an unobserved discrete-valued state variable (for example, 1 for a recession and 0 for an expansion) that is driven by a Markov chain. The transition from one state to another is governed by a matrix of transition probabilities that may be estimated from past data. The essence of this method thus is that the future will in some sense be like the past. A simple example of this idea is the regime-switching AR(1) model:

$$x_t = a_{1s_t}x_{t-1} + \varepsilon_t, \varepsilon_t \sim NID(0, \sigma^2) \quad (34)$$

where the regime  $s_t$  is the outcome of an unobserved two-state Markov chain with  $s_t$  independent of  $\varepsilon_\tau$  for all  $t$  and  $\tau$ . In this model, the time-varying slope parameter will take on different values depending on the state  $s$ . Once the model has been estimated by maximum likelihood methods, it is possible to infer how likely a given regime is to have generated the observed data at date  $t$ . An excellent review of the literature on hidden Markov models is provided by Cappé, Moulines and Ryden (2005); for a general treatment of state space representations of nonlinear models, also see Durbin and Koopman (2001).

The idea of *smooth transition regression models* is based on the observation that many economic variables are sluggish and will not move until some state variable exceeds a certain threshold. For example, price arbitrage in markets

will only set in once the expected profit of a trade exceeds the transaction cost. This observation has led to the development of models with fixed thresholds that depend on some observable state variable. Smooth transition models allow for the possibility that this transition occurs not all of a sudden at a fixed threshold but gradually, as one would expect in time series data that have been aggregated across many market participants. A simple example is the smooth-transition AR(1) model:

$$x_t = \Phi(z_{t-1}, \dots, z_{t-d}, \Gamma)x_{t-1} + \varepsilon_t \quad \varepsilon_t \sim NID(0, \sigma^2) \quad (35)$$

where  $\Phi(\cdot)$  denotes the transition function,  $z_t$  is a zero mean state variable denoting the current deviation of  $x_t$  from a (possibly time-varying) equilibrium level and  $\Gamma$  is the vector of transition parameters. Common choices for the transition function are the logistic or the exponential function. For example, we may specify  $\Phi(\cdot) = (\exp\{\gamma(z_{t-1})^2\})$  with  $\gamma < 0$ . If  $z_{t-1} = 0$ ,  $\Phi(\cdot) = 1$  and the model in (35) reduces to a random walk model; otherwise,  $\Phi(\cdot) < 1$  and the model in (35) reduces to a stationary AR(1). The degree of mean reversion is increasing in the deviation from equilibrium. For further discussion see Granger and Teräsvirta (1993).

## 6.2. Nonlinear dynamics in the conditional variance

While the preceding examples focused on nonlinear dynamics in the conditional mean, nonlinearities may also arise in higher moments. The leading example is the conditional variance. Many economic and financial time series are characterized by volatility clustering. Often interest centers on predicting these *volatility dynamics* rather than the conditional mean. The basic idea of modelling and forecasting volatility was set out in Engle's (1982) path-breaking paper on autoregressive conditional heteroskedasticity (ARCH). Subsequently, Bollerslev (1986) introduced the class of generalized autoregressive conditionally heteroskedastic (GARCH). Consider a decomposition of  $x_t$  into the one-step ahead conditional mean,  $\mu_{t|t-1} \equiv E(x_t|\Omega_{t-1})$ , and conditional variance,  $\sigma_{t|t-1}^2 \equiv Var(x_t|\Omega_{t-1})$ , where  $\Omega_{t-1}$  denotes the information set at  $t-1$ :

$$x_t = \mu_{t|t-1} + \sigma_{t|t-1}v_t \quad v_t \sim NID(0, 1) \quad (36)$$

The leading example of a GARCH model of the conditional variance is the GARCH(1,1) model, which is defined by the recursive relationship

$$\sigma_{t|t-1}^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1|t-2}^2 \quad (37)$$

where  $\varepsilon_t \equiv \sigma_{t|t-1}v_t$ , and the parameter restrictions  $\omega > 0$ ,  $\alpha \geq 0$ ,  $\beta \geq 0$  ensure that the conditional variance remains positive for all realizations of  $v_t$ . The standard estimation method is maximum likelihood. The basic GARCH(1,1) model may be extended to include higher-order lags, to allow the distribution of  $v_t$  to have fat tails, to allow for asymmetries in the volatility dynamics, to permit the conditional variance to affect the conditional mean, and to allow volatility shocks to have permanent effects or volatility to have long memory. It may also be extended to the multivariate case.

It follows directly from the formulation of the GARCH(1,1) model that the optimal, in the MMSE sense, one-step-ahead forecast equals  $\sigma_{t+1|t}^2$ . Similar expressions for longer horizons may be obtained by recursive updating. There is a direct link from the arrival of news to volatility measures and from volatility forecasts to risk assessments. These and alternative volatility models and the uses of volatility forecasts are surveyed in Andersen et al. (2006). For a comparison of GARCH models with the related and complementary

class of stochastic volatility models, see Andersen, Bollerslev and Diebold (2006) and Shephard (2005).

## 7. Applications

Time series analytic methods have many applications in economics. Here we consider five: (1) analysis of the cyclic properties of economic time series, (2) description of seasonality and seasonal adjustment, (3) forecasting, (4) dynamic econometric modelling, and (5) structural vector autoregressions.

### 7.1. Analysis of the cyclic properties of economic time series

Suppose that the time series  $\{x_t\}$  is a linearly non-deterministic stationary series and that the series  $\{y_t\}$  is formed from  $\{x_t\}$  by the linear operator

$$y_t = \sum_{j=m}^n w_j x_{t-j}, \quad \sum_m^n w_j^2 < \infty. \quad (38)$$

Such an operator is called a time-invariant linear filter. Analysis of the properties of such filters plays an important role in time series analysis since many methods of trend estimation or removal and seasonal adjustment may be represented or approximated by such filters. An interesting example that illustrates the potential pitfalls of using such filters is provided by Adelman (1965), who showed that the 20-year long swings in various economic series found by Kuznets (1961) may well have been the result of the trend filtering operations used in preliminary processing of the data. For a fuller treatment see Nerlove, Grether and Carvalho (1979, pp. 53–7).

Since the 1980s, there has been increased interest in the use of nonlinear filters for extracting the business cycle component of macroeconomic time series. Examples include the band-pass filter (Christiano and Fitzgerald, 2003) and the Hodrick–Prescott (HP) filter (Hodrick and Prescott, 1997; Ravn and Uhlig, 2002). The latter approach postulates that  $y_t = \tau_t + c_t$ , where  $\tau_t$  denotes the trend component and  $c_t$  the deviation from trend or ‘cyclical’ component of the time series  $y_t$ . The trend component is chosen to minimize the loss function:

$$\sum_{t=1}^T c_t^2 + \lambda \sum_{t=1}^T [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \quad (39)$$

where  $c_t = y_t - \tau_t$  and  $\lambda$  is a pre-specified parameter that depends on the frequency of the observations. The trade-off in this optimization problem is between the degree to which the trend component fits the data and the smoothness of the trend.

### 7.2. Description of seasonality and seasonal adjustment

Many economic time series exhibit fluctuations which are periodic within a year or a fraction thereof. The proper treatment of such seasonality, whether stochastic or deterministic, is the subject of a large literature, summarized rather selectively in Nerlove, Grether and Carvalho (1979, ch. 1). More recent treatments can be found in Hylleberg (1992), Franses (1996) and Ghysels and Osborn (2001).

Seasonality may be modelled and its presence detected using spectral analysis (Nerlove, 1964) or using time domain methods. Deterministic sea-

sonality, in the form of model parameters that vary deterministically with the season, offers no great conceptual problems but many practical ones. Stochastic seasonality is often modelled in the form of seasonal unit roots. In that case, seasonal differencing of the data removes the unit root component. Multiple time series may exhibit seasonal co-integration. Sometimes it is convenient to specify stochastic seasonality in the form of an UC model (Grether and Nerlove, 1970). Appropriate UC models may be determined directly or by fitting an ARIMA model and deriving a related UC model by imposing sufficient a priori restrictions (Hillmer and Tiao, 1982; Bell and Hillmer, 1984).

### 7.3. Forecasting

One of the simplest forecasting procedures for time series is exponential smoothing based on the relationship

$$\hat{x}_{t+1|t} = (1 - \theta)x_t + \theta\hat{x}_{t|t-1} \quad (40)$$

where  $x_t$  is the observed series and  $\hat{x}_{j|k}$  is the forecast of the series at time  $j$  made on the basis of information available up to time  $k$ . Muth (1960) showed that (40) provides an MMSE forecast if the model generating the time series is  $x_t - x_{t-1} = \varepsilon_t - \theta\varepsilon_{t-1}$ . Holt (1957) and Winters (1960) generalized the exponential smoothing approach to models containing more complex trend and seasonal components. Further generalization and proofs of optimality are contained in Theil and Wage (1964) and Nerlove and Wage (1964).

Perhaps the most popular approach to forecasting time series is based on ARIMA models of time series processes (Box and Jenkins, 1970). The developments discussed in the preceding paragraph led to the development of UC models, which give rise to restricted ARIMA model forms (Nerlove, Grether and Carvalho, 1979). State-space representations of these models permit the application of the Kalman filter to both estimation and forecasting. Harvey (1984) presents a unified synthesis of the various methods.

More recently, the focus has shifted from traditional forecasting methods towards methods that exploit the increased availability of a large number of potential predictors. Consider the problem of forecasting  $y_{t+h|t}$  based on its own current and past values as well as those of  $N$  additional variables,  $x_t$ . Of particular interest is the case in which the number of predictors,  $N$ , exceeds the number of time series observations,  $T$ . In that case, principal components analysis provides a convenient way of extracting a low-dimensional vector of common factors from the original data-set  $x_t$  (Stock and Watson, 2002a; 2002b). Forecasts that incorporate estimated common factors have proved successful in many cases in reducing forecast errors relative to traditional time series forecasting methods. Boivin and Ng (2005) provide a systematic comparison of alternative *factor model forecasts*. Another promising forecasting method is *Bayesian model averaging* across alternative forecasting models (Raftery, Madigan and Hoeting, 1997). The latter method builds on the literature on forecast combinations (Bates and Granger, 1969).

### 7.4. Dynamic econometric modelling

There is a close connection between multivariate time-series models and the structural, reduced and final forms of dynamic econometric models; the standard simultaneous-equations model (SEM) is a specific and restricted case.

Suppose that a vector of observed variables  $y_t$  may be subdivided into two classes of variables, ‘exogenous’,  $\{x_t\}$ , and endogenous,  $\{z_t\}$ . A dynamic, multivariate simultaneous linear system may be written.

$$\begin{bmatrix} \Psi_{11}(L) & \Psi_{12}(L) \\ 0 & \Psi_{22}(L) \end{bmatrix} \begin{pmatrix} z_t \\ x_t \end{pmatrix} = \begin{bmatrix} \Theta_{11}(L) & 0 \\ 0 & \Theta_{22}(L) \end{bmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} \quad (41)$$

where  $\Psi_{ij}(\cdot)$  and  $\Theta_{ij}(\cdot)$ ,  $i, j = 1, 2$  are matrix polynomials in the lag operator  $L$ . Such systems are known as vector ARMAX models and conditions for their identification are given by Hatanaka (1975). The reduced form of the system is obtained by expressing  $z_t$  as a function of lagged endogenous and current and lagged exogenous variables. The final form is then obtained by eliminating the lagged endogenous variables (see Zellner and Palm, 1974; Wallis, 1977).

### 7.5. Structural vector autoregressions

An important special case of the dynamic SEM is the structural vector autoregressive model in which all variables are presumed endogenous, the lag structure is unrestricted up to some order  $p$ , and identification of the structural form is achieved by imposing restrictions on the correlation structure of the structural innovations (Sims, 1980). The most common form of the structural VAR( $p$ ) model imposes restrictions on the contemporaneous interaction of structural innovations. Consider the structural form for a  $K$ -dimensional vector  $\{x_t\}$ ,  $t = 1, \dots, T$ :

$$B_0 x_t = \sum_{i=1}^p B_i x_{t-i} + \eta_t, \quad (42)$$

where  $\eta_t \sim (0, \Sigma_\eta)$  denotes the  $(K \times 1)$  vector of serially uncorrelated structural innovations (or shocks) and  $B_i$ ,  $i = 0, \dots, p$ , the  $(K \times K)$  coefficient matrices. Without loss of generality, let  $\Sigma_\eta = I$ . The corresponding reduced form is

$$x_t = \sum_{i=1}^p B_0^{-1} B_i x_{t-i} + B_0^{-1} \eta_t = \sum_{i=1}^p A_i x_{t-i} + \varepsilon_t \quad (43)$$

where  $\varepsilon_t \sim (0, \Sigma_\varepsilon)$ . Since  $\varepsilon_t = B_0^{-1} \eta_t$ , it follows that  $\Sigma_\varepsilon = B_0^{-1} B_0^{-1'}$ . Given a consistent estimate of the reduced form parameters  $A_i$ ,  $i = 1, \dots, p$ , and  $\Sigma_\varepsilon$ , the elements of  $B_0^{-1}$  will be exactly identified after imposing  $K(K-1)/2$  restrictions on the parameters of  $B_0^{-1}$  that reflect the presumed structure of the economy. Given estimates of  $B_0^{-1}$  and  $A_i$ ,  $i = 1, \dots, p$ , estimates of the remaining structural parameters may be recovered from  $B_i = B_0 A_i$ .

In practice, the number of restrictions that can be economically motivated may be smaller or larger than  $K(K-1)/2$ . Alternative estimation strategies that remain valid in the over-identified case include the generalized method of moments (Bernanke, 1986) and maximum likelihood (Sims, 1986). An instrumental variable interpretation of VAR estimation is discussed in Shapiro and Watson (1988). Semi-structural VAR models that are only partially identified have been proposed by Bernanke and Mihov (1998).

Alternative identification strategies may involve putting restrictions on the long-run behaviour of economic variables (Blanchard and Quah, 1989; King et al., 1991) or on the sign and/or shape of the impulse responses (Faust, 1998). Other possibilities include identification via heteroskedasticity (Rigobon, 2003) or the use of high-frequency data (Faust, Swanson and Wright, 2004).

The estimates of the structural VAR form may be used to compute the dynamic responses of the endogenous variables to a given structural shock, variance decompositions that measure the average contribution of each structural shock to the overall variability of the data, and historical decompositions of the path of  $x_t$  based on the contribution of each structural shock.

## 8. Conclusions

The literature on time series analysis has made considerable strides since the 1980s. The advances have been conceptual, theoretical and methodological. The increased availability of inexpensive personal computers in particular has revolutionized the implementation of time series techniques by shifting the emphasis from closed-form analytic solutions towards numerical and simulation methods. The ongoing improvements in information technology, broadly defined to include not only processing speed but also data collection and storage capabilities, are likely to transform the field even further. For example, the increased availability of large cross-sections of time series data, the introduction of ultra high-frequency data, the electronic collection of micro-level time series data (such as web-based data or scanner data), and the increased availability of data in real time all are creating new applications and spurring interest in the development of new methods of time series analysis. These developments already have brought together the fields of empirical finance and time series econometrics, resulting in the emergence of the new and fertile field of financial econometrics. As the use of time series methods becomes more widespread in applied fields, there will be increasing interest in the development of methods that can be adapted to the specific objectives of the end user. Another question of growing importance is how to deal with rapidly evolving economic environments in the form of structural breaks and other model instabilities. Finally, the improvement of structural time series models for macroeconomic policy analysis will remain a central task if time series analysis is to retain its importance for economic policy-making.

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### See also

[cointegration](#);  
[factor models](#);  
[forecasting](#);  
[long memory models](#);  
[maximum likelihood](#);  
[nonlinear time series analysis](#);  
[prediction](#);  
[seasonal adjustment](#);  
[spectral analysis](#);  
[spurious regression](#);  
[trend/cycle decomposition](#);  
[vector autoregressions](#).

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**Index terms**

ARFIMA models  
ARIMA models  
ARMA models  
ARMAX models  
autocovariance generating functions  
autoregressive conditional heteroskedasticity (ARCH)  
band-pass filter  
co-integration  
common factors  
Cournot, A.  
ergodicity and non-ergodicity  
estimation  
factor model forecasts  
forecasting  
generalized autoregressive conditionally heteroskedastic (GARCH)  
generalized method of moments  
Granger, C.  
heteroskedasticity  
Hodrick–Prescott (HP) filter  
inference  
Jevons, W.  
Kalman filter  
least squares  
linear processes  
long memory models  
Markov chain methods  
maximum likelihood  
minimum mean-square error (MMSE) criterion  
multiple time series analysis  
noise  
nonlinear time series analysis  
optimal prediction and extraction theory  
prediction  
principal components analysis  
regime switching models  
seasonal adjustment  
simultaneous-equations model  
Slutsky, E.  
smooth transition regression models  
specification  
spectral analysis  
spurious regressions  
state-space methods  
stationarity  
stochastic volatility models  
structural vector autoregressions  
time domain analysis  
time series analysis  
unit root  
unobserved components models  
vector autoregressions  
volatility dynamics

Wiener–Kolmogorov theory  
Wold decomposition theorem  
Yule, G.

**Index terms not found:**

ARFIMA models  
ergodicity and non-ergodicity  
generalized autoregressive conditionally heteroskedastic (GARCH)  
Markov chain methods  
nonlinear time series analysis  
optimal prediction and extraction theory  
spurious regressions  
structural vector autoregressions  
unobserved components models